

# Aperiodic Ising model on the Bethe lattice: Exact results

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(Dated: September 23, 2008)

We consider the Ising model on the Bethe lattice with aperiodic modulation of the couplings, which has been studied numerically in Phys. Rev. E **77**, 041113 (2008). Here we present a relevance-irrelevance criterion and solve the critical behavior exactly for marginal aperiodic sequences. We present analytical formulas for the continuously varying critical exponents and discuss a relationship with the (surface) critical behavior of the aperiodic quantum Ising chain.

PACS numbers: 64.60.-i, 64.10.+h, 05.50.+q

## I. INTRODUCTION

Disorder and different types of inhomogeneities are inevitable features of real materials. Their presence may modify the physical properties of a system and their effect can be particularly strong close to singularities, such as at phase transition points [1, 2]. In some cases the perturbation can change the universality class of a second-order phase transition. In this respect relevance or irrelevance of an inhomogeneous perturbation can be analyzed in terms of linear stability at the pure system's fixed point as first performed by Harris [3] for uncorrelated bond disorder. The classification of the critical behaviors of disordered systems with a random fixed point is a challenging and theoretically very difficult task.

Another type of inhomogeneities is introduced by quasiperiodic or, more generally, aperiodic modulations of the couplings. Since the discovery of quasicrystals [4] and due to the progress in molecular beam epitaxy, allowing for the preparation of good quality multilayers with a prescribed aperiodic structure [5], there has been an increased interest to study theoretically the phase transitions in such nonperiodic systems [6]. These systems can be considered as somehow intermediate between pure and random ones and are expected to display a rich variety of critical behaviors. Indeed a generalization of the Harris criterion predicts that, depending on the strength of the fluctuations of the aperiodic sequence and the value of the correlation length critical exponent  $\nu$  of the pure system, an aperiodic perturbation may be irrelevant, marginal, or relevant [7, 8]. A series of works on the critical behavior in different aperiodic systems confirms the validity of the generalized relevance-irrelevance criterion [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21].

Interestingly in the presence of aperiodicity one can observe truly marginal behavior. Then the critical exponents are nonuniversal and their value varies contin-

uously with the amplitude of the aperiodic perturbation. Such a behavior has been obtained exactly for the aperiodic Ising quantum chains using a renormalization group transformation or a finite-size scaling analysis [14, 15, 17, 18, 19, 20], as well as for the interface delocalization transition in the Penrose quasiperiodic lattice [13]. Nonuniversal critical behavior is expected to occur in real higher-dimensional systems, too, for example, in a three-dimensional tricritical system where  $\nu = 1/2$ . In this case, however, no exact results are available yet. Numerical studies of the related mean-field model with Fibonacci modulation of the couplings show nonuniversal critical behavior [22, 23].

More recently the Bethe-lattice Ising model, which also belongs to the mean-field universality class [24], has been studied numerically for two types of perturbations [25]. With a Fibonacci modulation of the couplings, the classical mean-field exponents are recovered whereas, for a period-doubling (PD) modulation, the magnetic exponents are nonuniversal. The difference in the relevance of the Fibonacci modulation for the two mean-field models is due to the different ways in which the mean-field behavior is realized, but this question has not been studied so far.

In this paper we continue the study of the aperiodic Bethe-lattice Ising model. Our motivations are twofold: First, we are interested in the formulation of a relevance-irrelevance criterion adapted to this system in order to explain the conflicting results of previous numerical works. Second, and our more important motivation, we can provide an *exact solution* of the problem and in this way we obtain analytical formulas for the continuously varying critical exponents, among others for the PD sequence studied before numerically.

The structure of the paper is the following. The model, the aperiodic sequences and the corresponding relevance-irrelevance criterion are presented in Sec. II. The critical behavior of marginal aperiodic sequences is studied analytically in Sec. III, in which we point out a close relationship with the (surface) critical properties of aperiodic quantum Ising chains. Our results are discussed in Sec. IV and details about the calculation of sums of

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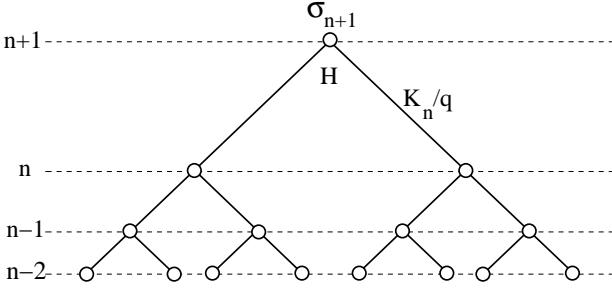


FIG. 1: Aperiodic Bethe lattice with coordination number  $z = q + 1 = 3$ . The couplings  $K_n/q$  are aperiodically modulated.

aperiodic variables are given in the Appendix.

## II. APERIODIC PERTURBATION AND ITS RELEVANCE

### A. Hamiltonian

We consider the spin 1/2 Ising model on a Bethe lattice with coordination number  $z = q + 1$ . The Hamiltonian reads

$$-\beta\mathcal{H} = \sum_n \sum_{(i,j) \in n} \frac{K_n}{q} \sigma_i \sigma_j + H \sum_i \sigma_i, \quad (1)$$

where  $\beta = 1/k_B T$  is the inverse temperature. The first sum runs over the successive layers of the lattice indexed by  $n$  (see Fig. 1) and the second over the bonds between the sites in layers  $n + 1$  and  $n$ . The couplings  $K_n/q$  are aperiodically modulated and properly normalized in order to allow us to take the mean-field limit  $q \rightarrow \infty$ . They are parametrized as

$$K_n = Kr^{f_n}, \quad f_n = 0, 1, \quad (2)$$

where  $r$  is the ratio of perturbed to unperturbed couplings and the binary variables  $f_n$  follow some aperiodic sequence.

Let  $\mathcal{O}_n = \langle \mathcal{O}_i \rangle$  be the thermal average of some local operator  $\mathcal{O}_i$  in layer  $n$ . This average is fluctuating from layer to layer due to the aperiodic modulation of the couplings. Following Ref. [25], a mean value  $\overline{\mathcal{O}}$  is defined by giving the same weight to the different layers,

$$\overline{\mathcal{O}} = \frac{1}{N} \sum_{n=1}^N \mathcal{O}_n. \quad (3)$$

With this choice, the usual mean-field critical behavior is obtained for the unperturbed system. When each layer is weighted according to its number of spins, the surface spins dominate the critical behavior which is then quite

unusual [26]: There is no long-range order at  $T > 0$  and the free energy displays a power-law singularity in  $H$  with a critical exponent increasing smoothly from 1 to infinity as  $T$  goes from 0 to the Bethe-Peierls temperature.

The Bethe lattice may be embedded in a  $d = \infty$  hypercubic lattice by placing each link into a different lattice direction. Then the geometrical distance  $L$ , measured in lattice parameter units, between two spins located  $N$  layers apart grows as [27]

$$L^2 = N \quad (4)$$

since the  $L$  steps are mutually orthogonal.

### B. Aperiodic sequences

As in Ref. [25] we consider aperiodic sequences generated via substitutions on the binary digits 0 and 1. For the PD sequence [28] we have  $\mathcal{S}(0) = 01$  and  $\mathcal{S}(1) = 00$  which, starting on 0, give successively

$$\begin{aligned} &0, \\ &01, \\ &0100, \\ &01000101. \end{aligned} \quad (5)$$

The properties of the sequence can be deduced from its substitution matrix [29, 30] with entries  $M_{ij}$  giving the numbers  $n_i^{\mathcal{S}(j)}$  of digits of type  $i$  in  $\mathcal{S}(j)$ . In the case of the PD sequence one obtains

$$M = \begin{pmatrix} n_0^{\mathcal{S}(0)} & n_0^{\mathcal{S}(1)} \\ n_1^{\mathcal{S}(0)} & n_1^{\mathcal{S}(1)} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}. \quad (6)$$

The entries in  $M^p$  give the numbers of digits of each type in the sequence after  $p$  iterations. The length  $N$  of the sequence obtained after  $p$  iterations (which is also the number of layers on the Bethe lattice) is related to the leading eigenvalue  $\Lambda_1$  of the substitution matrix through  $N = \Lambda_1^p$ . Let

$$n_N = \sum_{n=1}^N f_n, \quad \rho_\infty = \lim_{N \rightarrow \infty} \frac{n_N}{N}, \quad (7)$$

be the number of 1 in a sequence with  $N$  digits and the corresponding asymptotic density, respectively. On the Bethe lattice, according to Eq. (3), the mean value of the coupling is given by

$$\begin{aligned} \overline{K} &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N K_n = K + \lim_{N \rightarrow \infty} \frac{n_N}{N} K(r-1) \\ &= K + \rho_\infty \Delta, \end{aligned} \quad (8)$$

where  $\Delta = K(r-1)$  is the amplitude of the aperiodic modulation of the couplings. The mean deviation from

$\overline{K}$  on a system with  $N \sim \Lambda_1^p$  layers takes the form

$$\begin{aligned}\overline{\delta K}(N) &= \frac{1}{N} \sum_{n=1}^N (K_n - \overline{K}) = \Delta \left( \frac{n_N}{N} - \rho_\infty \right) \\ &\sim \frac{\Delta}{N} \Lambda_2^p \sim \Delta N^{\omega-1},\end{aligned}\quad (9)$$

where  $\omega$  is the wandering exponent of the sequence given by

$$\omega = \frac{\ln |\Lambda_2|}{\ln \Lambda_1}, \quad (10)$$

in terms of the second leading eigenvalue  $\Lambda_2$  of the substitution matrix. For the PD sequence, according to Eq. (6), we have  $\Lambda_1 = 2$ ,  $\Lambda_2 = -1$ , so that  $\omega = 0$ .

### C. Relevance-irrelevance criterion

The Harris argument [3], showing that thermal randomness is a relevant perturbation only when the specific heat exponent  $\alpha$  of the pure system is positive, has been generalized to the case of aperiodic perturbations in Refs. [7, 8].

The argument can be adapted to our problem as follows: Near the critical point of the pure system, the relevant length is the correlation length  $\xi$  diverging as  $t^{-\nu}$  where  $t \sim |K - K_c|$  measures the deviation from the critical temperature. To the length  $\xi$  is associated the number of layers  $N = \xi^2$  according to Eq. (4) and the aperiodic perturbation induces a shift in the critical temperature  $\overline{\delta t} \sim \overline{\delta K}(N)$  which, according to Eq. (9), takes the form  $\overline{\delta t} \sim \xi^{2(\omega-1)} \sim t^{-2\nu(\omega-1)}$ . The ratio

$$\frac{\overline{\delta t}}{t} \sim t^{-\phi}, \quad \phi = 1 + 2\nu(\omega - 1) \quad (11)$$

gives the relative strength of the aperiodic perturbation. It diverges, and thus the perturbation is relevant, when the crossover exponent  $\phi > 0$ . It is irrelevant when  $\phi < 0$  and marginal when  $\phi = 0$ . In this latter case the aperiodicity may lead to a nonuniversal behavior with some exponents varying continuously with the amplitude of the perturbation.

The same result can be obtain by studying the scaling behavior of the perturbation amplitude  $\Delta$  in Eq. (9). Under a change of the length scale by a factor  $b = L/L'$ , Eq. (4) leads to  $N' = N/b^2$  and  $\overline{\delta K}(N)$ , with scaling dimension  $y_t = 1/\nu$ , and transforms as

$$(\overline{\delta K})' \sim \Delta' N'^{\omega-1} = b_t^y \Delta N^{\omega-1} \quad (12)$$

so that

$$\Delta' = b_t^{y_t+2(\omega-1)} \Delta. \quad (13)$$

Thus the scaling dimension of  $\Delta$  is  $\phi/\nu$  and the perturbation grows under rescaling (is relevant) when  $\phi$  is positive.

A continuous variation of the magnetic exponents was observed in Ref. [25] for the PD sequence with  $\omega = 0$ . This marginal behavior is expected since  $y_t = 1/\nu = 2$  for the mean-field Ising model. On the contrary, the Fibonacci sequence, with  $\omega = -1$  [15], leads to an irrelevant perturbation. It does not change the critical behavior which remains classical.

## III. CRITICAL BEHAVIOR

### A. Finite-size behavior of the magnetization

We consider an  $n+1$ -generation branch defined as an initial site with spin  $\sigma_{n+1}$  connected to  $q = z - 1$   $n$ -generation branches as shown in Fig. 1; a one-generation branch is a single site. Let  $Z_n^\pm$  be the sum of the contributions to the partition function of an  $n$ -generation branch with initial spin either up (+) or down (-). It satisfies the recursion relation

$$Z_{n+1}^\pm = e^{\pm H} \left( e^{\pm K_n/q} Z_n^+ + e^{\mp K_n/q} Z_n^- \right)^q. \quad (14)$$

In the mean-field limit  $q \rightarrow \infty$ , the  $n$ th layer magnetization  $m_n = \langle \sigma_n \rangle$  may be written as

$$m_n = \frac{Z_n^+ - Z_n^-}{Z_n^+ + Z_n^-} \quad (15)$$

since the contribution of the single branch going forward can be neglected compared to the contributions of the  $q$  branches going backward. Expanding the exponentials in Eq. (14) one obtains

$$Z_{n+1}^\pm = e^{\pm H} \left[ Z_n^+ + Z_n^- \pm \frac{K_n}{q} (Z_n^+ - Z_n^-) + O\left(\frac{K_n^2}{q^2}\right) \right]^q \quad (16)$$

so that

$$\begin{aligned}\lim_{q \rightarrow \infty} \frac{Z_{n+1}^\pm}{(Z_n^+ + Z_n^-)^q} &= e^{\pm H} \lim_{q \rightarrow \infty} \left[ 1 \pm \frac{K_n}{q} m_n + O\left(\frac{K_n^2}{q^2}\right) \right]^q \\ &= \exp[\pm(H + K_n m_n)],\end{aligned}\quad (17)$$

and the layer magnetization satisfies the recursion relation

$$m_{n+1} = \tanh(H + K_n m_n). \quad (18)$$

Expanding to the first order in the external field  $H$  and to the third order in the first layer magnetization  $m_1$ , one has

$$\begin{aligned}m_{n+1} &= H \prod_{i=1}^n K_i \sum_{k=1}^n \prod_{j=1}^k K_j^{-1} + m_1 \prod_{i=1}^n K_i \\ &\quad - \frac{m_1^3}{3} \prod_{i=1}^n K_i \sum_{k=1}^n \prod_{j=1}^k K_j^2 + \dots.\end{aligned}\quad (19)$$

According to Eq. (3) the mean value of the magnetization on a system with  $n$  layers is given by

$$\bar{m} = \frac{1}{N} \sum_{n=1}^N m_n = a_0 H + a_1 m_1 - \frac{a_3}{3} m_1^3 + \dots \quad (20)$$

with

$$\begin{aligned} a_0 &= \frac{1}{N} \sum_{n=1}^{N-1} \prod_{i=1}^n K_i \sum_{k=1}^n \prod_{j=1}^k K_j^{-1}, \\ a_1 &= \frac{1}{N} \left( 1 + \sum_{n=1}^{N-1} \prod_{i=1}^n K_i \right), \\ a_3 &= \frac{1}{N} \sum_{n=1}^{N-1} \prod_{i=1}^n K_i \sum_{k=1}^n \prod_{j=1}^k K_j^2. \end{aligned} \quad (21)$$

The critical point of the system is obtained by analyzing the asymptotic behavior of  $a_1$ . It is divergent (goes to zero), if  $\left(\prod_{j=1}^N K_j\right)^{1/N}$  is greater (smaller) than 1. Consequently the critical point is given by the condition

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N \ln K_j = \overline{\ln K} = 0, \quad (22)$$

leading to

$$K_c = r^{-\rho_\infty}. \quad (23)$$

## B. Relation with the one-dimensional Ising model in a transverse field

Let us consider the inhomogeneous quantum Ising chain with Hamiltonian

$$H = - \sum_l J_l s_l^z s_{l+1}^z - h \sum_l s_l^x, \quad (24)$$

where  $s_l^x$  and  $s_l^z$  are the components of a Pauli spin operator associated with site  $l$ ,  $J_l$  is the first-neighbor exchange interaction, and  $h$  is the transverse field. On a chain with size  $L$  and the end spin fixed, the surface magnetization  $m_s$  satisfies the relation [31]

$$m_s^{-2} = 1 + \sum_{l=1}^L \prod_{i=1}^l \lambda_i^{-2}, \quad (25)$$

where  $\lambda_i = J_i/h$ . This is just the form of the sum giving  $N a_1$  in Eq. (21). At the critical coupling, generally given by  $\overline{\ln \lambda} = 0$  [32],  $m_s^{-2}$  scales like  $L^{2x_s}$  where  $x_s = \beta_s/\nu$  is the scaling dimension of the surface magnetization with  $x_s = 1/2$  for the unperturbed quantum Ising chain.

This exponent has been determined analytically in the case of marginal aperiodic modulations of the couplings

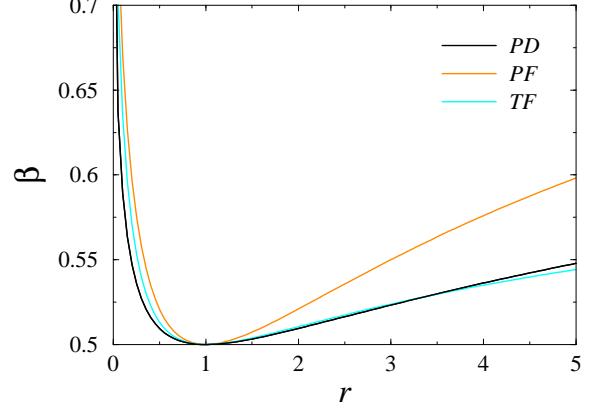


FIG. 2: (Color online) Variation of the critical exponent  $\beta$  of the spontaneous magnetization with the coupling ratio  $r$  for the PD, paper-folding (PF), and three-folding (TF) sequences. The exponent is minimum and takes its mean-field value  $\beta = 1/2$  for the unperturbed system at  $r = 1$ .

[14, 15, 17, 18]. Since  $N$  is replaced by  $L$ ,  $2\nu = 1$  is replaced by  $\nu_{\text{Ising}} = 1$  in the expression (11) of the crossover exponent  $\phi$ . Thus marginal behavior is obtained for the same value  $\omega = 0$  of the wandering exponent.

With the parametrization  $\lambda_i = \lambda r^{f_i}$ , one obtains

$$x_s(r) = \frac{\ln(r^{1/3} + r^{-1/3})}{2 \ln 2} \quad (26)$$

for the PD sequence. Changing  $r$  into  $r^{-1}$  does not change  $x_s$  for this sequence, but this is not generally true.

The leading behaviors of the sums appearing in Eq. (21) are calculated in the Appendix. One may notice that since  $K_i$  corresponds to  $\lambda_i^{-2}$ ,  $r$  in Eq. (26) has to be replaced by  $r^{-1/2}$  in the scaling exponent of  $a_1$ . At the critical point, the different coefficients scale with  $N$  as follows

$$\begin{aligned} a_0 &\sim N^{x_0}, & x_0 &= 2x_s(r^{1/2}) + 2x_s(r^{-1/2}) - 1, \\ a_1 &\sim N^{x_1}, & x_1 &= 2x_s(r^{-1/2}) - 1, \\ a_3 &\sim N^{x_3}, & x_3 &= 2x_s(r^{-1}) + 2x_s(r^{-1/2}) - 1. \end{aligned} \quad (27)$$

## C. Finite-size scaling and critical exponents

In a finite-size system with  $N$  layers the free energy density  $\bar{g}$  is a function of the deviation from the critical temperature  $t$  and the external field  $H$ , of the system size  $N$  and also of the magnetization  $m_1$  of the first layer. Under a change of the length scale by a factor  $b$ , these variables transforms as

$$t' = b^{y_t} t, \quad H' = b^{y_H} H, \quad N' = \frac{N}{b^2}, \quad m'_1 = b^{2\tau} m_1. \quad (28)$$

For a truly marginal system, the thermal dimension  $y_t = 1/\nu$  has to keep its unperturbed value  $y_t = 2$ .

The critical exponents  $\beta$ , governing the temperature dependence of the spontaneous magnetization, and  $\gamma$ ,

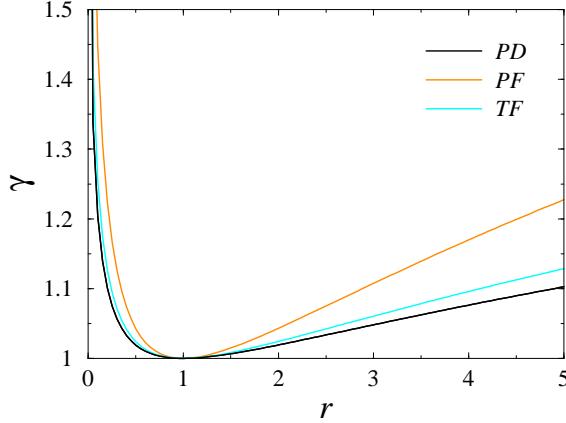


FIG. 3: (Color online) Variation of the critical exponent  $\gamma$  of the susceptibility with the coupling ratio  $r$  for the PD, PF, and TF sequences. The exponent is minimum and takes its mean-field value  $\gamma = 1$  for the unperturbed system at  $r = 1$ .

governing the temperature dependence of the susceptibility, have the following expressions

$$\beta = \frac{d_c - y_H}{y_t}, \quad \gamma = \frac{2y_H - d_c}{y_t}, \quad y_t = 2, \quad (29)$$

where  $d_c$  is the upper critical dimension of the problem. The dimensions  $y_H$ ,  $\tau$ , and  $d_c$  remain to be determined for the aperiodic system.

The free energy density transforms as

$$\bar{g}(t, H, N, m_1) = b^{-d_c} \bar{g} \left( b^{y_t} t, b^{y_H} H, \frac{N}{b^2}, b^{2\tau} m_1 \right). \quad (30)$$

The mean value of the magnetization  $\bar{m} = \partial \bar{g} / \partial H$  reads

$$\bar{m}(t, H, N, m_1) = b^{-d_c + y_H} \bar{m} \left( b^{y_t} t, b^{y_H} H, \frac{N}{b^2}, b^{2\tau} m_1 \right). \quad (31)$$

It can be expanded in powers of  $m_1$  as

$$\begin{aligned} \bar{m}(t, H, N, m_1) &= \sum_{k \geq 0} b^{y_H - d_c + 2k\tau} \\ &\times \bar{m}^{(k)} \left( b^{y_t} t, b^{y_H} H, \frac{N}{b^2} \right) m_1^k. \end{aligned} \quad (32)$$

At the critical temperature, the leading dependence on a small external field  $H$  comes from the term of order zero in the expansion (32). With  $t = 0$  and  $b^2 = N$  we have

$$\begin{aligned} \bar{m}(0, H, N, m_1) &= N^{(y_H - d_c)/2} \varphi_H^{(0)}(N^{y_H/2} H) + O(m_1) \\ &\sim N^{(2y_H - d_c)/2} H. \end{aligned} \quad (33)$$

When  $H = 0$ ,  $\bar{m}$  is odd in  $m_1$  so that, with  $b^2 = N$ , one obtains

$$\begin{aligned} \bar{m}(t, 0, N, m_1) &= N^{\tau + (y_H - d_c)/2} \varphi_t^{(1)}(Nt) m_1 \\ &+ N^{3\tau + (y_H - d_c)/2} \varphi_t^{(3)}(Nt) m_1^3 + \dots \end{aligned} \quad (34)$$

TABLE I: Critical behavior of the Bethe lattice with an aperiodic modulation following the PD sequence: comparison of the exact values of the critical exponents  $\beta$ ,  $\gamma$  in Eq. (35) and  $\delta$  deduced from the Widom scaling law to the numerical values obtained in Ref. [25].

	$r = 1$	$r = 2$	$r = 7$
$\beta$ (exact)	1/2	0.5094795	0.5675859
$\beta$ (numerical)		0.5093(4)	0.5664(5)
$\gamma$ (exact)	1	1.0192114	1.1491583
$\gamma$ (numerical)		1.0197(2)	1.1499(4)
$\delta$ (exact)	3	3.0004955	3.0246421
$\delta$ (numerical)		3.0006(2)	3.0266(9)

Comparing Eqs. (33) and (34) to Eqs. (20) and (27) one can deduce the values of the critical exponents

$$\begin{aligned} \gamma &= x_0 = 2x_s(r^{1/2}) + 2x_s(r^{-1/2}) - 1, \\ \beta &= \frac{x_3 - 3x_1}{2} = x_s(r^{-1}) - 2x_s(r^{-1/2}) + 1, \\ \tau &= \frac{x_3 - x_1}{2} = x_s(r^{-1}), \\ d_c &= 2 + 4x_s(r^{-1}) + 4x_s(r^{1/2}) - 4x_s(r^{-1/2}). \end{aligned} \quad (35)$$

For the unperturbed system ( $r = 1$ ,  $x_s = 1/2$ ) the mean-field Ising values,  $\gamma = 1$ ,  $\beta = 1/2$ ,  $d_c = 4$ , are recovered. The variations of  $\beta$  and  $\gamma$  with  $r$  for the PD sequence are shown in Figs. 2 and 3. Similar results for the paper-folding (PF) and three-folding (TF) sequences are also shown [33]. In these cases the values [17, 18]

$$x_s(r) = \frac{\ln(1 + r^{-1})}{2 \ln 2} \text{ (PF)}, \quad x_s(r) = \frac{\ln(2 + r)}{2 \ln 3} \text{ (TF)}. \quad (36)$$

have been used in Eq. (35).

For the PD sequence, the values of  $\beta$ ,  $\gamma$ , and  $\delta$  (deduced from the Widom scaling law  $\delta = 1 + \gamma/\beta$ ) given in Table I for  $r = 2$  and  $r = 7$  compare well with the numerical values obtained in Ref. [25] (notice that  $r$  in the present work corresponds to  $r - 1$  in [25]).

#### IV. DISCUSSION

In this paper we have studied the critical behavior of the Ising model on a Bethe lattice with aperiodic modulation of the couplings. Our first result is the relevance-irrelevance criterion of Eq. (11) which is adapted to the Bethe-lattice problem. As for the aperiodic quantum Ising chain or previous mean-field models, it has a form typical of a one-dimensional aperiodicity. The difference lies in the fact that the length of the sequence does not scale here like a physical length  $L$  but like the number of layers  $N$  on the Bethe lattice, which itself scales like  $L^2$ . As a consequence, the correlation length exponent  $\nu$  is replaced by  $2\nu$ . Thus for the Bethe-lattice problem

with  $\nu = 1/2$  the aperiodicity is marginal when the wandering exponent  $\omega = 0$  as for the Ising quantum chain with  $\nu_{\text{Ising}} = 1$ . For the same reason, the aperiodicity is irrelevant on the Bethe lattice for the Fibonacci sequence with  $\omega = -1$  [25] whereas it is marginal for the one-dimensional mean-field models [22, 23].

We have solved the critical properties of the aperiodic Bethe-lattice Ising model exactly and we have observed further similarities with the quantum Ising chain. For marginal aperiodic sequences, such as the PD sequence, the critical exponents are nonuniversal in both cases and the Bethe-lattice exponents can be expressed in terms of the surface magnetization exponent of the quantum Ising chain, taken at different values of the aperiodic coupling ratio  $r$ . Since for the quantum Ising chain there is a vast literature about exact solutions for different marginal sequences, from these we can immediately translate the corresponding analytical results for the Bethe lattice.

We have also noticed that for the Bethe lattice, in order to satisfy the scaling relations, a varying upper critical dimension  $d_c(r)$  has to be introduced. As a matter of fact, this result follows from an analysis of the Ginzburg criterion [34], too. Something similar occurs for the Ising quantum chain, in which the dynamical exponent  $z$  was found to be  $r$  dependent [16, 17]. As a consequence, here also the effective dimension of the system  $d = 1 + z$  varies continuously with  $r$ .

Our investigation could be extended into several directions. For relevant aperiodic sequences, such as the Rudin-Shapiro sequence, first-order transition is expected in one range of the ratio,  $r < 1$ , whereas in the other range,  $r > 1$ , the magnetization should exhibit an essential singularity at the critical point [35], instead of a power law observed for marginal perturbations. This type of essential singular behavior is probably the rule for random interactions, too.

### Acknowledgments

This work has been supported by the Hungarian National Research Fund under Grants No. OTKA TO48721, No. K62588, No. K75324, and No. MO45596. The Laboratoire de Physique des Matériaux is Unité Mixte de Recherche CNRS No. 7556.

### APPENDIX: CALCULATION OF THE SUMS THROUGH RENORMALIZATION

Let us consider the sum

$$S_N(K, r) = 1 + \sum_{p=1}^N \prod_{i=1}^p Kr^{f_i} = \sum_{p=0}^N K^p r^{n_p}, \quad n_0 = 0, \quad (\text{A.1})$$

such that  $a_1$  in Eq. (21) is given by  $S_{N-1}(K, r)/N$ . For the PD sequence the following relations are satisfied (see

Ref. [14] where 0 and 1 are exchanged)

$$f_{2k} = 1 - f_k, \quad f_{2k+1} = 0, \quad n_{2k} = n_{2k+1} = k - n_k. \quad (\text{A.2})$$

Splitting the sum into even and odd parts and ignoring minor end corrections, one obtains

$$\begin{aligned} S_N(K, r) &= \sum_{k=0}^{N/2} K^{2k} r^{n_{2k}} + \sum_{k=0}^{N/2} K^{2k+1} r^{n_{2k+1}} \\ &= \sum_{k=0}^{N/2} (K^2 r)^k r^{-n_k} + K \sum_{k=0}^{N/2} (K^2 r)^k r^{-n_k} \\ &= (1 + K) S_{N/2}(K^2 r, r^{-1}). \end{aligned} \quad (\text{A.3})$$

A second iteration leads to

$$S_N(K, r) = (1 + K)(1 + K^2 r) S_{N/4}(K^4 r, r) \quad (\text{A.4})$$

which is a renormalization transformation leaving  $r$  invariant, dividing  $N$  by 4 and changing  $K$  into  $K' = K^4 r$ . In the infinite system this transformation has a nontrivial fixed point at

$$K^* = K_c = r^{-1/3} \quad (\text{A.5})$$

which is the critical coupling of the problem, in agreement with the general result of Eq. (23). At the critical point, Eq. (A4) gives the finite-size scaling relation

$$\begin{aligned} S_N(K_c, r) &= (1 + r^{-1/3})(1 + r^{1/3}) S_{N/4}(K_c, r) \\ &= (r^{1/6} + r^{-1/6}) S_{N/4}(K_c, r). \end{aligned} \quad (\text{A.6})$$

Injecting the power law  $S_N(K_c, r) \simeq AN^{\omega(r)}$  into Eq. (A6), one finally obtains

$$\omega(r) = \frac{2 \ln(r^{1/6} + r^{-1/6})}{2 \ln 2} = 2x_s(r^{-1/2}) \quad (\text{A.7})$$

in agreement with the value of  $x_1$  given in Eq. (27).

The values of  $Na_0$  and  $Na_3$  in Eq. (21) are given by the sum

$$T_N(K, r; m) = \sum_{p=1}^N K^p r^{n_p} \sum_{l=1}^p K^{ml} r^{mn_l} \quad (\text{A.8})$$

with  $m = -1$  for  $a_0$  and  $m = 2$  for  $a_3$ . The leading contribution to this sum for large  $N$  values is given by

$$\begin{aligned} T_N(K, r; m) &\simeq \sum_{p=0}^N K^p r^{n_p} S_p(K^m, r^m) \\ &\simeq A \sum_{p=0}^N K^p r^{n_p} p^{\omega(r^m)}. \end{aligned} \quad (\text{A.9})$$

Proceeding as before for  $S_N$ , the sum can be split into even and odd parts and after two iterations one obtains [36]

$$T_N(K_c, r; m) \simeq 4^{\omega(r^m)} (r^{1/6} + r^{-1/6}) T_{N/4}(K_c, r; m). \quad (\text{A.10})$$

It follows that, at the critical point, this sum scales with  $N$  as

$$\begin{aligned} T_N(K_c, r; m) &\sim N^{\omega(r; m)}, \quad \omega(r; m) = \omega(r) + \omega(r^m) \\ \omega(r; m) &= 2x_s(r^{-1/2}) + 2x_s(r^{-m/2}). \end{aligned} \quad (\text{A.11})$$

With the appropriate values of  $m$  one recovers the exponents in Eq. (27).

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